APPENDIX A

Fourier Series for Some Common Waveforms

FOURIER SERIES

The Fourier series for a periodic function $f(t)$ can be expressed in trigonometric form as

$$f(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

where
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

Sines and cosines of the same frequency can be combined into one sinusoid, resulting in an alternative expression for a Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

where $$C_n = \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

or

$$f(t) = a_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega_0 t + \theta_n)$$

where $$C_n = \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad \theta_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

The rms value of $f(t)$ can be computed from the Fourier series.

$$F_{\text{rms}} = \sqrt{\sum_{n=0}^{\infty} F_{n,\text{rms}}^2} = \sqrt{a_0^2 + \sum_{n=1}^{\infty}\left(\frac{C_n}{\sqrt{2}}\right)^2}$$

HALF-WAVE RECTIFIED SINUSOID (FIG. A-1)

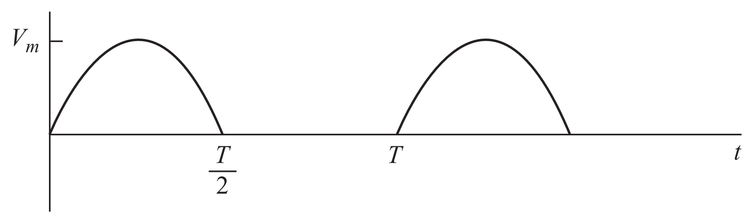

Figure A-1 Half-wave rectified sine wave.

$$v(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\sin(\omega_0 t) - \sum_{n=2,4,6\ldots}^{\infty} \frac{2V_m}{(n^2-1)\pi}\cos(n\omega_0 t)$$

FULL-WAVE RECTIFIED SINUSOID (FIG. A-2)

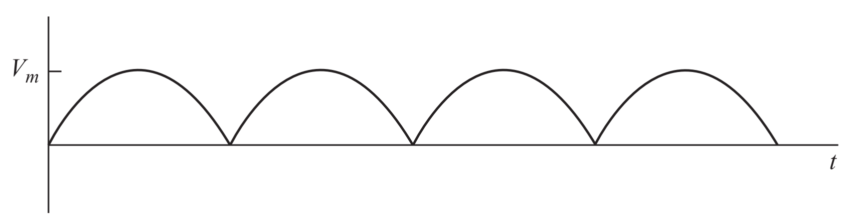

Figure A-2 Full-wave rectified sine wave.

$$v_o(t) = V_o + \sum_{n=2,4,\ldots}^{\infty} V_n \cos(n\omega_0 t + \pi)$$

where $$V_o = \frac{2V_m}{\pi}$$

and $$V_n = \frac{2V_m}{\pi}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

THREE-PHASE BRIDGE RECTIFIER (FIG. A-3)

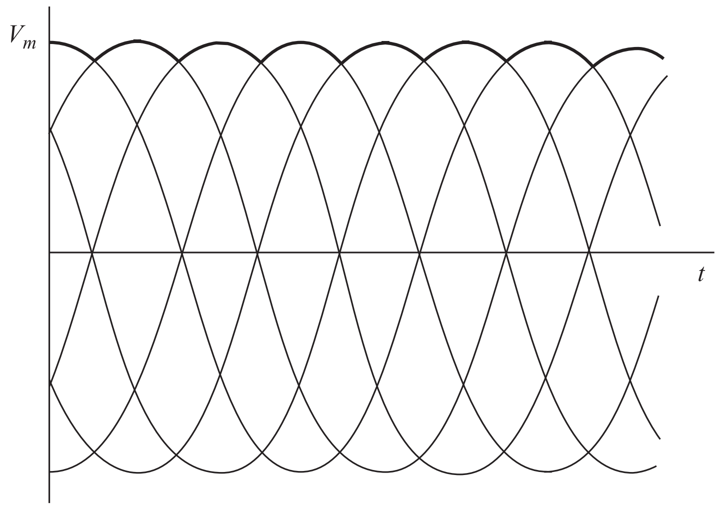

Figure A-3 Three-phase six-pulse bridge rectifier output.

The Fourier series for a six-pulse converter is

$$v_o(t) = V_o + \sum_{n=6,12,18,\ldots}^{\infty} V_n \cos(n\omega_0 t + \pi)$$

$$V_o = \frac{3V_{m,\mathrm{L-L}}}{\pi} = 0.955\ V_{m,\mathrm{L-L}}$$

$$V_n = \frac{6V_{m,\mathrm{L-L}}}{\pi(n^2-1)} \qquad n = 6, 12, 18, \ldots$$

where $V_{m,\mathrm{L-L}}$ is the peak line-to-line voltage of the three-phase source, which is $\sqrt{2}V_{\mathrm{L-L,rms}}$.

The Fourier series of the currents in phase *a* of the ac line (see Fig. 4-17) is

$$i_a(t) = \frac{2\sqrt{3}}{\pi} I_o \left(\cos\omega_0 t - \frac{1}{5}\cos 5\omega_0 t + \frac{1}{7}\cos 7\omega_0 t - \frac{1}{11}\cos 11\omega_0 t + \frac{1}{13}\cos 13\omega_0 t - \ldots\right)$$

which consists of terms at the fundamental frequency of the ac system and harmonics of order $6k \pm 1$, $k = 1, 2, 3, \ldots$.

PULSED WAVEFORM (FIG. A-4)

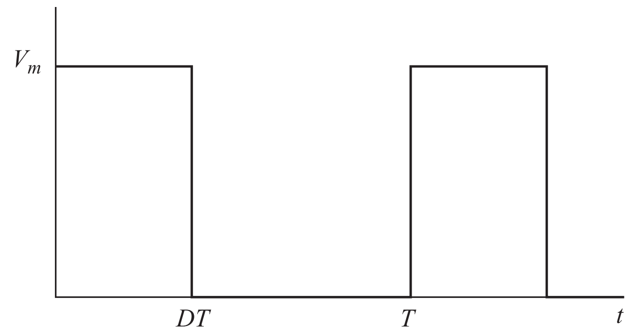

Figure A-4 A pulsed waveform.

$$a_0 = V_m D$$

$$a_n = \left(\frac{V_m}{n\pi}\right)\sin(n2\pi D)$$

$$b_n = \left(\frac{V_m}{n\pi}\right)[1 - \cos(n2\pi D)]$$

$$C_n = \left(\frac{\sqrt{2}V_m}{n\pi}\right)\sqrt{1 - \cos(n2\pi D)}$$

SQUARE WAVE (FIG. A-5)

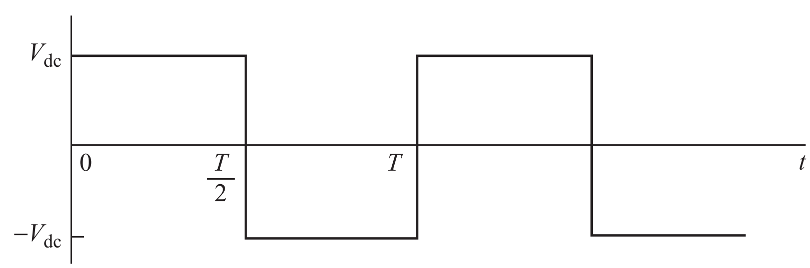

Figure A-5 Square wave.

The Fourier series contains the odd harmonics and can be represented as

$$v_o(t) = \sum_{n\ \text{odd}} \left(\frac{4V_{dc}}{n\pi}\right)\sin(n\omega_0 t)$$

MODIFIED SQUARE WAVE (FIG. A-6)

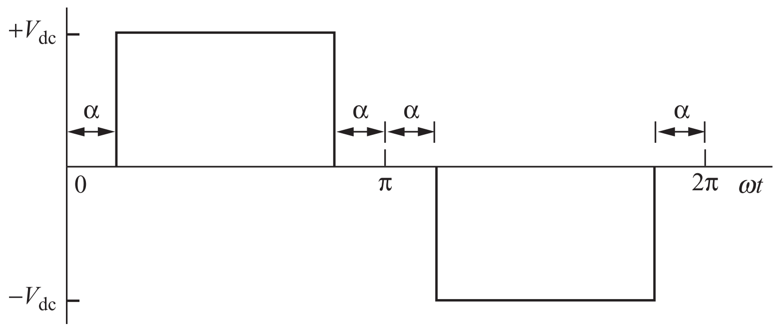

Figure A-6 A modified square wave.

The Fourier series of the waveform is expressed as

$$v_o(t) = \sum_{n \text{ odd}} V_n \sin(n\omega_0 t)$$

Taking advantage of half-wave symmetry, the amplitudes are

$$V_n = \left(\frac{4V_{dc}}{n\pi}\right)\cos(n\alpha)$$

where α is the angle of zero voltage on each end of the pulse.

THREE-PHASE SIX-STEP INVERTER (FIG. A-7)

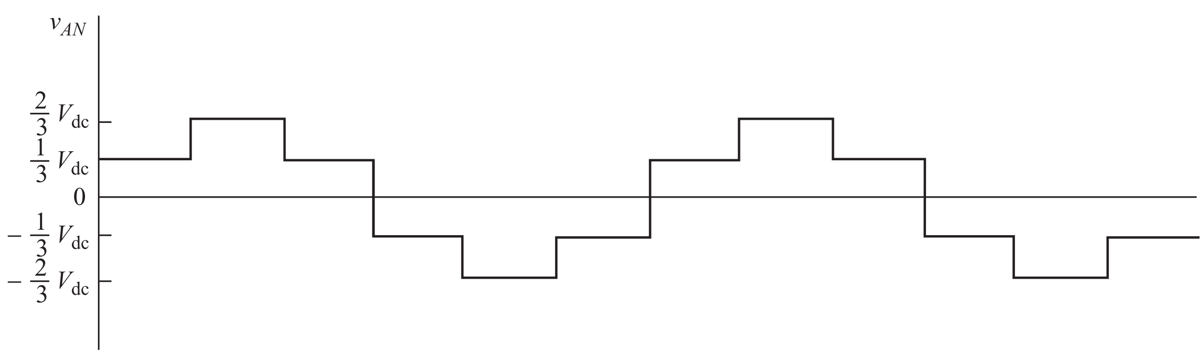

Figure A-7 Three-phase six-step inverter output.

The Fourier series for the output voltage has a fundamental frequency equal to the switching frequency. Harmonic frequencies are of order $6k \pm 1$ for $k = 1, 2, \ldots$ ($n = 5, 7, 11, 13, \ldots$). The third harmonic and multiples of the third do not exist, and even harmonics do not exist. For an input voltage of V_{dc}, the output for an ungrounded wye-connected load (see Fig. 8-17) has the following Fourier coefficients:

$$V_{n,\mathrm{L-L}} = \left| \frac{4V_{dc}}{n\pi} \cos\left(n\frac{\pi}{6} \right) \right|$$

$$V_{n,\mathrm{L-N}} = \left| \frac{2V_{dc}}{3n\pi} \left[2 + \cos\left(n\frac{\pi}{3} \right) - \cos\left(n\frac{2\pi}{3} \right) \right] \right| \qquad n = 1,5,7,11,13,\ldots$$